Propagation Constants of Circular Cylindrical Waveguides Containing Ferrites*

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Summary—The paper describes some results of a theoretical and experimental investigation of the propagation behavior of circular cylindrical waveguides containing longitudinally magnetized ferrite rods. As long as no concentration of the RF-magnetic field in the ferrite occurs, theoretical expressions for the propagation constants can be given by applying first-order perturbation method. Faraday rotation measurements have been made between 5000 and 7600-mcs using commercially available ferrites. Reasonable agreement between theoretical and experimental results has been found for a thin axial ferrite rod in an air-filled guide in both cases of saturated and nonsaturated ferrites. Energy concentration in the ferrite determines the propagation behavior in the partially filled waveguide. This effect can be enhanced by surrounding the ferrite rod with a dielectric tube. For a given rod diameter and permittivity of the tube there is an optimum outer diameter of the tube for which the Faraday-rotation becomes maximum.

Introduction

THE first microwave applications of ferrites used a round waveguide containing an axial ferrite pencil in an axial magnetic field. The microwave Faraday effect was demonstrated by Hogan[1] in 1952, and a number of waveguide components have been developed which utilize this phenomenon. Such Faraday rotation devices continue to be of great practical importance.

In this paper some results are described of a theoretical and experimental investigation of circular waveguides containing longitudinally magnetized ferrites. One aim is to provide an improved and more quantitative understanding of the propagation behavior of this type of waveguide structure and thereby aid the design of Faraday rotation wave guide components.

Another aim is to find out how such devices can be optimized; *e.g.*, for high-speed switches or modulators the main problem is to obtain rapid build-up and decay of magnetizing current. To achieve this, it is necessary to use a minimum number of turns on the magnetizing coil. Furthermore the demagnetizing field should be kept as small as possible. The application of long thin ferrite rods having small demagnetization factors leads necessarily to longitudinal field structures, *i.e.* Faraday rotation devices. To get along with low magnetizing current the Faraday rotation for a given magnetic field intensity should be optimum.

The special cases of

1) a circular waveguide filled with ferrite,
2) a circular waveguide filled with an axial ferrite rod or tube and dielectric of the same permittivity as the ferrite, and
3) a circular waveguide filled with a thin axial ferrite rod or tube and dielectric of different permittivity

are investigated in detail, because these are the only cases where explicit theoretical expressions for the angle of rotation can be obtained so far. By means of these results the general case of the partially filled waveguide can also be understood and its behavior estimated within certain limits of accuracy sufficient for many practical cases. These considerations are confirmed by experimental results, where the angle of rotation as function of the rod diameter is measured for magnetized and saturated ferrites.

For a partially ferrite-loaded guide, the surrounding medium of dielectric can effect both an increase and a decrease of the angle of rotation depending on the diameter of the ferrite rod, the dimensions of the surrounding dielectric tube and its permittivity. In a detailed experimental investigation the influence of these parameters has been studied. For a certain ferrite rod and a certain permittivity of the surrounding dielectric tube there is always an optimum diameter for the tube.

Longitudinally Magnetized Ferrites in Circular Waveguides

The fact that the permeability is a scalar quantity for circularly polarized fields in an infinite ferrite medium means that the normal modes of propagation along the direction of magnetization are right and left circularly polarized waves. For the considerably more complicated structure of a round rod of ferrite enclosed axially by a circular waveguide carrying the dominant wave, it may still be true that circularly polarized waves are the normal modes.

The waveguide Faraday effect consists of the rotation of the whole field pattern of the "linearly polarized" TE_{11}-wave, where the diametral plane of maximum electric field intensity is defined as plane of polarization. Faraday rotation of an angle θ means a rotation of the plane of polarization and consequently of the whole field pattern by that angle. In the round guide two linearly polarized TE_{11}-waves can exist in independent states of polarization, those planes being perpendicular to each other. Because of this twofold degeneracy there are also circularly polarized TE_{11}-waves possible.

For a circular waveguide containing an axial ferrite rod of arbitrary radius and magnetization an extensive

* Received by the PGMTT, March 30, 1959. The work on this paper was performed at RCA Labs., Princeton, N. J., in 1956, during a leave of absence from the University of Goettingen.

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[1] C. L. Hogan, "The microwave gyrator," *Bell Sys. Tech. J.*, vol. 31, pp. 1–31; 1952.

analysis of the propagation behavior has been made by van Trier[2] and by Suhl and Walker.[3] These investigations show that wave forms with vanishing longitudinal component of E or H do not exist. Far from ferromagnetic resonance, however, this sixth field component is one order of magnitude smaller than the other components. Therefore, the mode spectrum may be divided into two groups, namely quasi-TE- and quasi-TM-modes, which become TE- and TM-waves when the anisotropy is gradually removed. For the dominant mode the following picture of Faraday rotation can be given. Suppose a circular cylinder of gyromagnetic medium extends along the axis of the guide for $z>0$ and a linearly polarized TE_{11}-wave is incident from $z=-\infty$. This wave may be decomposed mathematically into right and left circularly polarized TE_{11}-waves of equal amplitudes and propagation constants. The wave transmitted through the anisotropic medium will consist of two circularly polarized quasi-TE_{11}-waves. As long as the anisotropy is small, the amplitudes of the two components will be nearly the same, and the slight difference in the imaginary parts of the propagation constants will result in a Faraday rotation of the transmitted linearly polarized quasi-TE_{11}-wave. In the case of large anisotropies, however, real and imaginary parts of the propagation constants are different for the two circularly polarized waves, and therefore transmitted and reflected waves are no longer linearly polarized. By measuring the ellipticity of the transmitted wave it can be checked experimentally, whether the presumptions of the calculation are fulfilled by the ferrite loaded waveguide section under test.

Although the authors cited above[2,3] have obtained the general solution to the problem in form of its characteristic equation, owing to the complexity of this transcendental equation, no explicit expression for the propagation constants can be given. In order to avoid lengthy computing programs one may resort to techniques of approximation, *e.g.*, to a perturbation method. That takes as a starting point a situation, the propagation problem of which is solved, and the change in propagation constant due to a slight change in the original state of the system is calculated. For the problems under discussion here, the small change of the state means a slight modification of the permeability or permittivity within certain regions of the system. Such modifications may be caused by the weak magnetization of an originally unmagnetized ferrite filling the waveguide, or by the introduction of a slender pencil into the originally empty guide. The propagation constant for a magnetized ferrite rod of arbitrary radius, coaxial with a circular waveguide, the space between guide wall and rod being filled with an isotropic dielectric of the same permittivity as the ferrite, can also be calculated by means of perturbation theory. The discussion of these problems opens the understanding for the practically more important case of a ferrite rod of any radius in an air-filled guide.

Consider a waveguide with perfectly conducting walls enclosing a lossless medium of the tensor permeability $[\mu]$ and permittivity $[\epsilon]$. Let the cutoff frequency be ω_c and let the electromagnetic field of the corresponding mode be characterized by the vectors **e** and $\boldsymbol{h}$. Provided that the cutoff frequencies of other modes are not too close to ω_c a modification of the material constants $[\Delta\epsilon]\ll[\epsilon]$ and $(\Delta\mu]\ll[\mu]$ leads in first-order approximation to the change of cutoff frequency given by the expression[4,5]

$$-\frac{\Delta\omega_c}{\omega_c}\quad\frac{1}{2}\,\frac{\int_F\int\left\{[\Delta\epsilon]ee^* + Z^2[\Delta\mu]hh^*\right\}df}{Z^2\int_F\int[\mu]hh^*df}, \tag{1}$$

which follows from the equality of the time averages of stored electrical and magnetic energy. The integration is performed over the cross section F of the guide, and Z stands for the characteristic impedance of free space.

$$Z=\sqrt{\frac{\mu_0}{\epsilon_0}}=377\ [\text{ohms}].^6$$

Throughout this paper it is presumed that the circular waveguide fully or partially loaded with a coaxial ferrite rod will not propagate higher order modes, but carries only the dominant mode. For the unperturbed guide the components of the magnetic field of the circularly polarized TE_{11}-waves are

$$\left.\begin{aligned} h_z &= h_0 J_1\left(s_{11}\frac{r}{a}\right)\exp\left(i\omega t - kz \pm \phi\right)\\ h_r &= \mp\, ih_0\frac{\lambda_c}{\lambda_G}J_1'\left(s_{11}\frac{r}{a}\right)\exp\left(i\omega t - kz \pm \phi\right)\\ h_\phi &= h_0\frac{1}{2\pi}\frac{\lambda_c^2}{\lambda_G r}J_1\left(s_{11}\frac{r}{a}\right)\exp\left(i\omega t - kz \pm \phi\right)\end{aligned}\right\} \tag{2}$$

where the two different signs characterize the right or left circularly polarized wave, respectively. r, ϕ, z are right circular cylindrical coordinates and a is the radius of the waveguide. Furthermore, J_1 is the Bessel function of first order, $s_{11}=1.841$ equals the first root of its derivation $J_{1'}$. Guide wavelength λ_G, cutoff wavelength λ_c and vacuum wavelength λ are connected by

[2] A. A. T. M. van Trier, "Guided electromagnetic waves in anisotropic media," *Appl. Sci. Res.*, vol. 3 (B), pp. 305–371; 1953.

[3] H. Suhl and L. R. Walker, "Topics in guided wave propagation through gyromagnetic media," *Bell Sys. Tech. J.*, vol. 33, pp. 575–659, 939–986 and 1133–1194; 1954.

[4] J. O. Artman and P. E. Tannenwald, "Measurement of susceptibility tensor in ferrites, *J. Appl. Phys.*, vol. 26, pp. 1124–1132; 1955.

[5] H. Severin, "Effectiveness and limits of a first-order perturbation method for waveguides," to be published.

[6] The rationalized mks or Giorgi system of units is used throughout this paper.

$= 1.256\ 10^{-6}$ Vs/Am

$$\lambda_G = \frac{\lambda}{\sqrt{\epsilon\mu - \left(\frac{\lambda}{\lambda_c}\right)^2}}, \tag{3}$$

where for the TE_{11}-mode

$$\lambda_c = 3.412a.$$

For a circular waveguide filled with ferrite, (2) is correct as long as the ferrite is not magnetized and therefore behaves like an isotropic dielectric. With a dc-magnetic field applied longitudinally such that the occurring anisotropy is kept small, the slightly different propagation constants of the two circularly polarized quasi-TE_{11}-waves may be calculated in first order approximation using (1) and the unperturbed field (2).

When the ferrite is uniformly magnetized and saturated by a dc magnetic field H_z applied along the positive z-axis and subjected to an RF-field

$$h_x, h_y, h_z \ll H_z; \tag{4}$$

the permeability tensor $[\mu]$ relating the flux density $\boldsymbol{b}$ and the magnetic field intensity $\boldsymbol{h}$ according to

$$\boldsymbol{b} = \mu_0[\mu]\boldsymbol{h}$$

has the form[7]

$$[\mu] = \begin{bmatrix} \mu_1 & -i\kappa & 0 \\ i\kappa & \mu_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

where the tensor components μ_1 and κ are in general complex quantities because of the magnetic losses of the material. For right circular cylindrical coordinates the permeability tensor (5) is identical with that in Cartesian coordinates. This can be seen by applying rotation operators to the matrix equation (4) in Cartesian space and obtaining (4) in cylindrical coordinates. If the originally unmagnetized ferrite becomes weakly anisotropic by magnetization, then

$$[\Delta\mu]hh^* = (\mu_1 - 1)(|h_r|^2 + |h_\phi|^2) + 2_k\mathrm{Im}(h_r^* h_\phi)$$

and with (2)

$$[\Delta\mu]hh^* = h_0^2\left(\frac{\lambda_c}{\lambda_G}\right)^2 \left\{ (\mu_1 - 1)\left[F_1'^2\left(s_{11}\frac{r}{a}\right) + \frac{J_1^2\left(s_{11}\frac{r}{a}\right)}{\left(s_{11}\frac{r}{a}\right)^2} \right] \pm \pm 2\kappa J_1'\left(s_{11}\frac{r}{a}\right)\frac{J_1\left(s_{11}\frac{r}{a}\right)}{s_{11}\frac{r}{a}} \right\}. \tag{6}$$

[7] D. Polder, "On the theory of ferromagnetic resonance," *Phil. Mag.*, vol. 40, pp. 99–115; 1949.

The integrals occurring in (1) can be computed elementary. With the results given in Appendix I and

$$\frac{\Delta k_G}{k_G} = -\epsilon\mu\left(\frac{k}{k_G}\right)^2 \frac{\Delta\omega_c}{\omega_c},$$

from (3) it follows in first order approximation that

$$\frac{\Delta k_G}{k_G} = \frac{1}{2\pi}\left(\frac{k_c}{k_G}\right)^2\left(\frac{s_{11}}{a}\right)^2 \frac{1}{(s_{11}^2 - 1)J_1^2(s_{11})} \frac{1}{h_0^2}\cdot\iint [\Delta\mu]hh^* df \tag{7}$$

where the integration has to be performed over the cross section of the ferrite.

The angle of Faraday rotation is given by the difference in the phase velocities of the two circularly polarized components and the path length through the ferrite medium according to

$$\theta = \frac{l}{2}(k_G^- - k_G^+). \tag{8}$$

Together with (6) and (7) one gets

$$\theta = \frac{2\kappa}{(s_{11}^2 - 1)J_1^2(s_{11})} k_G l \cdot \int J_1\left(s_{11}\frac{r}{a}\right) J_1'\left(s_{11}\frac{r}{a}\right) d\left(s_{11}\frac{r}{a}\right). \tag{9}$$

Therefore in first order approximation Faraday rotation does not depend on the main diagonal component μ_1 of the permeability tensor but only on its off-diagonal component κ.

Various Ferrite Configurations in Circular Waveguides

The problems mentioned in the introduction will be treated in the following. It is supposed that the ferrite is magnetized longitudinally and saturated, and the operating frequency is far away from ferromagnetic resonance. Then the anisotropy is small enough to apply the perturbation method for calculating the change of propagation constants or the Faraday rotation, respectively, in first order approximation.

The angle of Faraday rotation, according to (8), may be found as the difference of the propagation constants k_G^- and k_G^+, given in Appendix II, or more directly by using (9). From it for the waveguide filled with ferrite it follows immediately that

$$\theta_0 = \frac{\kappa}{s_{11}^2 - 1} k_G l. \tag{10}$$

For plane electromagnetic waves in an unbound ferrite medium one has in the same approximation

$$\theta_\infty = \tfrac{1}{2}\kappa\sqrt{\epsilon}kl,$$

and therefore

$$\frac{\theta_0}{\theta_\infty} = \frac{2}{s_{11}^2 - 1} \frac{\lambda/\sqrt{\epsilon}}{\lambda_G} = 0.83 \frac{\lambda/\sqrt{\epsilon}}{\lambda_G},$$

which result is identical with that found by Suhl and Walker[8] in a more rigorous way. θ_0 is smaller than θ_∞ as to be expected, because the field is not circularly polarized at every point of the guide cross section. Furthermore with respect to the plane wave case the waveguide causes additional dispersion.

Another problem of some interest is that of an axial ferrite rod or tube of arbitrary radius and infinite length in a round waveguide, where the remainder of the waveguide is filled with a nonmagnetic dielectric, whose permittivity is equal to that of the ferrite. Although this problem may be not of immediate practical significance its solution will be helpful in treating the practically more important case of a ferrite rod in an air-filled guide. Because the field (2) is exact in the unmagnetized case, there is no restriction on the diameter of the ferrite rod. From (9) it follows that

$$\frac{\theta}{\theta_0} = \frac{J_1^2\left(s_{11}\frac{\rho}{a}\right)}{J_1^2(s_{11})} \tag{11}$$

for a ferrite rod of the diameter 2ρ, and

$$\frac{\theta}{\theta_0} = 1 - \frac{J_1^2\left(s_{11}\frac{\rho}{a}\right)}{J_1^2(s_{11})} \tag{12}$$

for the complementary tube of the inner diameter 2ρ and the outer diameter $2a$. Considering the rod, for the two limiting cases of very small or large diameter, respectively, from (11) one gets

$$\rho \ll a: \frac{\theta}{\theta_0} = \frac{1}{4} \frac{s_{11}^2}{J_1^2(s_{11})} \left(\frac{\rho}{a}\right)^2 \tag{11a}$$

$$\rho \approx a: \frac{\theta}{\theta_0} = 1 - (s_{11}^2 - 1)\left(\frac{a-\rho}{a}\right)^2. \tag{11b}$$

As regards the tube for very small or large wall thickness result (12) simplifies to

$$\rho \approx a: \frac{\theta}{\theta_0} = (s_{11}^2 - 1)\left(\frac{a-\rho}{a}\right)^2 \tag{12a}$$

$$\rho \ll a: \frac{\theta}{\theta_0} = 1 - \frac{1}{4} \frac{s_{11}^2}{J_1^2(s_{11})} \left(\frac{\rho}{a}\right)^2. \tag{12b}$$

In Figs. 1 and 2 the Faraday rotation computed according to (11) and (12) has been drawn as function of the diameter or wall thickness of the ferrite rod or tube, respectively. For comparing these two cases in Fig. 3

[8] H. Suhl and L. R. Walker, "Faraday rotation of guided waves," *Phys. Rev.*, vol. 86, pp. 122–123; 1952.

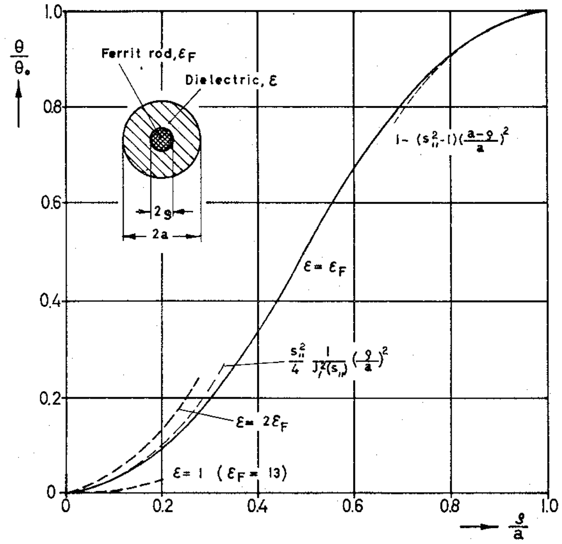

Fig. 1—Faraday rotation of a TE_{11}-wave propagating through a round waveguide containing an axial rod of ferrite embedded in a dielectric of the permittivity ϵ.

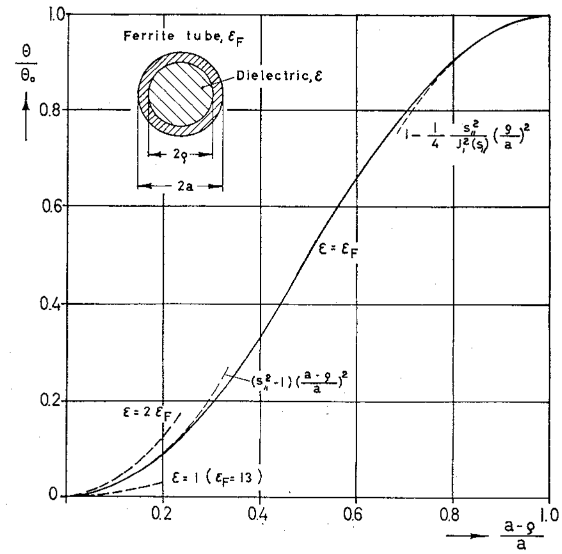

Fig. 2—Faraday rotation of a TE_{11}-wave propagating through a round waveguide containing a ferrite tube at its wall and a dielectric filler of the permittivity ϵ.

the angle of rotation has been plotted vs the area occupied by the ferrite. With equal areas (quantities) of ferrite the axial rod gives considerably larger rotation than the tube as to be expected from the h-field configuration of the TE_{11}-mode (Fig. 4).

The practically most important problem is that of a thin axial ferrite rod inserted axially into an air-filled guide. Because of the high permittivity of the ferrites (between 10 and 15) relative to that of air, rather severe restrictions are put on the diameter of the rods to which a perturbation method is applicable. This limitation would be substantially relaxed if exact solutions for dielectric rods of high permittivity introduced axially into round guides would be available, which could be used as the basis for magnetic perturbation calculation. Unfortunately, since such solutions are not known so

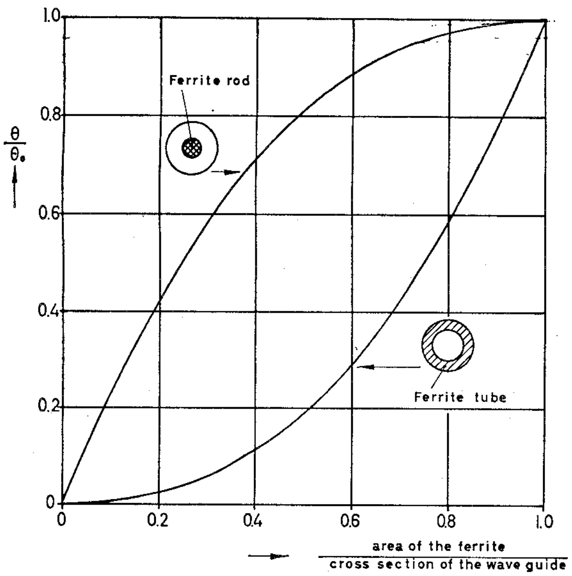

Fig. 3—Comparison of the Faraday rotation of a TE_{11}-wave propagating through a round waveguide containing an axial rod of ferrite with the rotation for a ferrite tube at the wall of the guide. The remaining part of the cross-section is filled with a dielectric of the same permittivity as the ferrite.

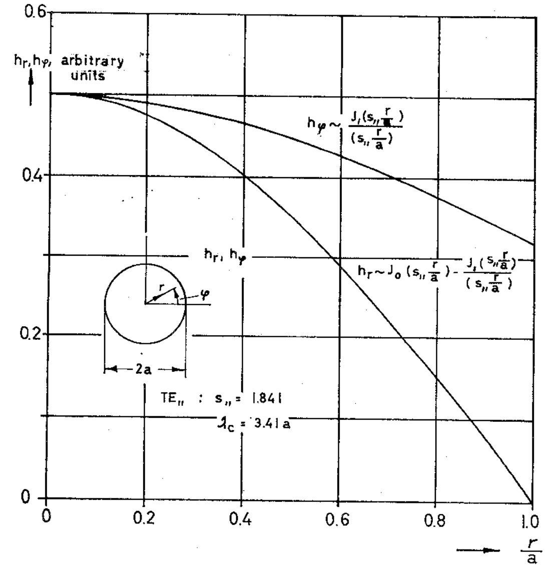

Fig. 4—Transverse components of the magnetic field of a circularly polarized TE_{11}-wave in a round waveguide. The same field distribution holds for a linearly polarized TE_{11}-wave along the diameter characterized by $\phi=\pi/4$.

far, the "unperturbed," *i.e.*, unmagnetized case of the dielectrically loaded guide may be treated independent of the magnetic perturbation also by first-order perturbation method.

As before, the radius of the guide is a and that of the ferrite rod is $\rho \ll a$. Its permittivity is ϵ_F, and we consider the more general case, that the remainder of the waveguide is filled with a dielectric of the permittivity ϵ. For $s_{11}(r/a) \ll 1$

$$J_1\left(s_{11}\frac{r}{a}\right) \approx \frac{1}{2}\, s_{11}\frac{r}{a}, \qquad J_1'\left(s_{11}\frac{r}{a}\right) \approx \frac{1}{2}$$

and from (9) it follows that

$$\theta = \frac{\kappa}{4}\,\frac{s_{11}^2}{(s_{11}^2-1)J_1^2(s_{11})}\, l k_G^{(\epsilon)} \left(\frac{\rho}{a}\right)^2$$

and together with (10) one has

$$\frac{\theta}{\theta_0} = \frac{1}{4}\,\frac{s_{11}^2}{J_1^2(s_{11})}\,\frac{k_G^{(\epsilon)}}{k_G^{(\epsilon_F)}} \left(\frac{\rho}{a}\right)^2. \tag{13}$$

As to be expected, this result becomes identical with (11a) if $\epsilon=\epsilon_F$. $k_G^{(\epsilon)}$ and $k_G^{(\epsilon_F)}$ are the propagation constants of the round waveguide filled with a dielectric of the permittivity ϵ or with an unmagnetized ferrite of the permittivity ϵ_F, respectively. Instead of $k_G^{(\epsilon)}$ one would expect the propagation constant k_G of the unmagnetized structure of ferrite rod and surrounding dielectric. A first order approximation of k_G has been given in Appendix II. In (13) from k_G only $k_G^{(\epsilon)}$ remains because terms of higher than quadratic order have to be neglected in this approximation.

If on the other hand the radius of the ferrite rod approaches the radius of the guide, *i.e.*, $s_{11}\,(r/a) \approx s_{11}$, with

$$J_1\left(s_{11}\frac{r}{a}\right) \approx J_1(s_{11}) - \frac{s_{11}^2-1}{2} J_1(s_{11}) \left(\frac{a-r}{a}\right)^2$$

$$J_1'\left(s_{11}\frac{r}{a}\right) \approx \frac{s_{11}^2-1}{s_{11}} J_1(s_{11})\,\frac{a-r}{a} + \frac{1}{2}\,\frac{s_{11}^2-3}{s_{11}} J_1(s_{11}) \left(\frac{a-r}{a}\right)^2$$

from (9) the result (11b) is obtained. The permittivity of the thin walled dielectric tube representing the perturbation in the waveguide otherwise filled with ferrite does not influence the Faraday rotation in first order approximation.

Finally, also for the two structures complementary to those treated above the Faraday rotation can be computed by first-order perturbation method. If the waveguide contains a thin-walled ferrite tube adjoining the wall of the guide and the remainder is filled with a dielectric of the permittivity ϵ, one finds

$$\frac{\theta}{\theta_0} = (s_{11}^2 - 1)\,\frac{k_G^{(\epsilon)}}{k_G^{(\epsilon_F)}} \left(\frac{a-\rho}{a}\right). \tag{14}$$

For the waveguide filled with ferrite and an axial dielectric rod of small radius $\rho \ll a$ and permittivity ϵ one obtains result (12b). Again the permittivity of the dielectric perturbation does not influence the Faraday rotation in first order approximation.

For the cases that the dielectric is air ($\epsilon=1$) or its permittivity is twice as high as that of the ferrite ($\epsilon_F=13$), the results (13) and (14) are drawn in Figs. 1 and 2 for comparison with the case where ferrite and dielectric have equal permittivities.

Experimental Results

The Faraday rotation of a few ferrites has been measured using a similar equipment as described by Hogan.[1]

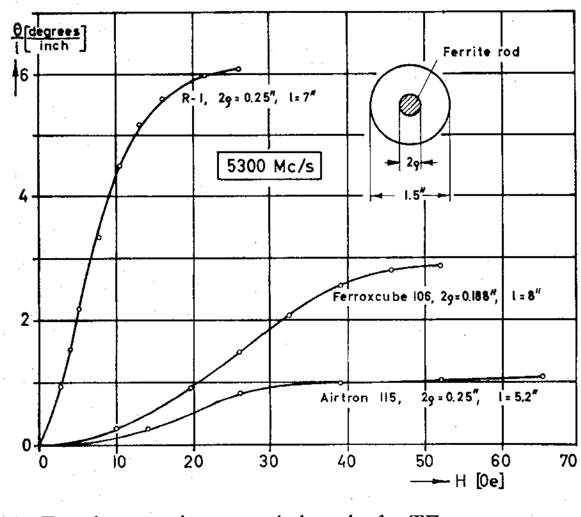

Fig. 5—Faraday rotation per unit length of a TE_{11}-wave propagating at a frequency of 5300 mc through a round waveguide containing an axial ferrite rod, as function of the applied longitudinal dc magnetic field.

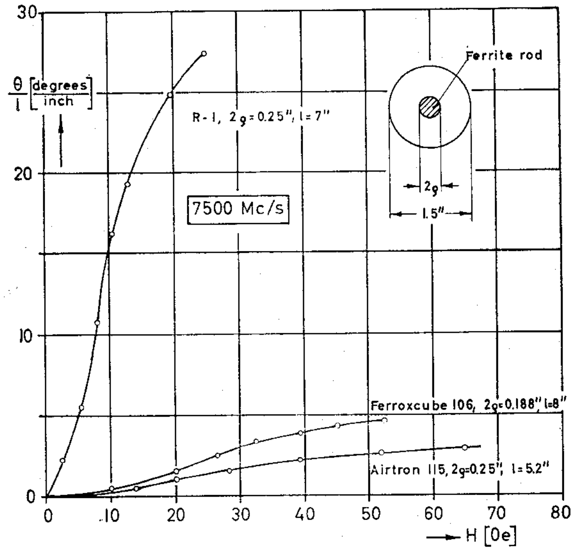

Fig. 6—Faraday rotation per unit length of a TE_{11}-wave propagating at a frequency of 7500 mc through a round waveguide containing an axial ferrite rod, as function of the applied longitudinal dc magnetic field.

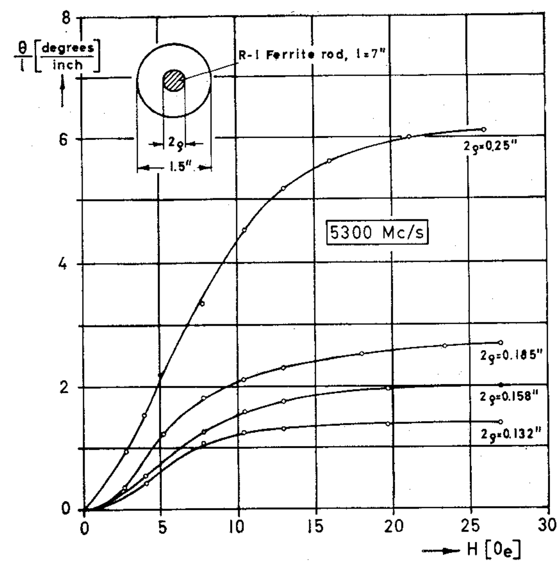

Fig. 7—Faraday rotation per unit length of a TE_{11}-wave propagating at a frequency of 5300 mc through a round waveguide containing an axial $R-1$ ferrite rod, as function of the applied longitudinal dc magnetic field for various diameters of the ferrite rod.

The ferrite samples are supported by two polyfoam discs and placed along the axis of a circularly cylindrical waveguide carrying the dominant TE_{11}-mode. This section is fed from a rectangular waveguide by means of a proper nonreflective transition. On the opposite end of the circular waveguide the analyzer consisting of another nonreflective transition and a rectangular waveguide with detector mouth is supported, so that it can be rotated around the longitudinal axis of the system. The circular guide is placed in a specially constructed coil system[9] producing an axial magnetic field of constant intensity along the length of the ferrite rod.

Measurements have been made at frequencies between 5 and 7.6 kmc on the following ferrite materials available to us in form of thin long rods:

General Ceramics	$R-1$	$(M_s=2150$ G)
Ferroxcube	106	$(M_s=3300$ G)
Airtron	115	$(M_s=1850$ G).

From Kittel's formula for the ferromagnetic resonance in a finite body[10] it follows that for a long thin pencil approximately

$$f_{\text{res}} = \frac{\gamma}{2\pi}\left(H_z + \frac{1}{2}\frac{M_s}{\mu_0}\right) \tag{15}$$

holds, where

$$\frac{\gamma}{2\pi} = 2.8\left(\frac{Mc/s}{Oe}\right).$$

Long thin specimens of soft ferrite materials magnetized longitudinally saturate at applied fields of about 10 to 50 oersteds. Therefore in (15) H_z may be neglected against $\frac{1}{2}M_s/\mu_0$, and the resonance frequency is determined by the saturation magnetization. For the materials under investigation ferromagnetic resonance must be expected at frequencies of 3, 4.6 or 2.6 mc, so that in any case the operating frequency was above ferromagnetic resonance.

For the various ferrites, Figs. 5–7 show the Faraday rotation measured as function of the applied field with rod diameter and frequency as parameters. The curves look very similar to the curve of magnetization vs applied field, as to be expected and observed by others.[1,2,11] According to (10) and (13) for the cases of

[9] H. Severin, "A coil system producing a uniform magnetic field along its axis," to be published.

[10] C. Kittel, "On the theory of ferromagnetic resonance absorption," *Phys. Rev.*, vol. 73, pp. 155–161; 1948.

[11] P. J. B. Clarricoats, A. G. Hayes and A. F. Harvey, "A survey of the theory and applications of ferrites at microwave frequencies," *Proc. IEE*, vol. 104, B Suppl., pp. 267–282; 1957.

the ferrite filled guide and slender axial ferrite pencils the rotation per unit path length is proportional to the off-diagonal component κ of the permeability tensor (5). This may be true in first order approximation also for rods of arbitrary diameter. Because of

$$\kappa = \frac{\gamma \dfrac{M_s}{\mu_0} \omega}{\omega^2 - \omega^2_{\text{res}}} \approx \frac{\gamma}{\mu_0} \frac{M_s}{\omega} \tag{16}$$

for $\omega \gg \omega_{\text{res}}$, the Faraday rotation in a saturated medium is proportional to the saturation magnetization. In the case of an unsaturated medium Rado[12] has shown that (16) remains valid when therein the saturation magnetization M_s is replaced by the magnetization M. In

$$M = \mu_0(\mu - 1)H_i$$

the internal field H_i may be replaced approximately by the applied field H_z for rods being long enough to keep the demagnetizing field negligibly small. Therefore the curve of Faraday-rotation as function of the applied magnetic field is very similar to the curve of magnetization vs applied field. Comparing corresponding curves in Figs. 5 and 6 one sees that the rotation increases with increasing frequency. For a waveguide partially loaded with ferrite, because of the difference in permittivities an increase in frequency causes the electromagnetic energy to become concentrated more and more in the ferrite rod, thus resulting in an increase of rotation. An increase of the rod diameter should have the same effect on the energy concentration in the ferrite, and this is confirmed by the measuring results shown in Fig. 7.

Those effects of energy concentration are of course not taken into account by the first-order perturbation method, which is based on the unperturbed fields of the empty guide. The results of the perturbation method must therefore be insufficient, if with increasing frequency or diameter of the ferrite rod energy concentration occurs. Its influence can clearly be seen from Fig. 8 by comparing the curves of Faraday rotation vs frequency for rods of different diameters. Such measurements indicate up to which rod diameter the application of first-order perturbation method may be possible. In the case of Fig. 8 this limit is $d = 0.132$ inch, possibly $d = 0.158$ inch for frequencies below 6000 mc. Fig. 9 indicates a remarkable agreement between measurements and calculation for rod diameters $d = 0.1$ inch and 0.125 inch. For an infinite medium the plane wave theory shows that Faraday-rotation is independent of frequency if the operating frequency is much higher than the ferromagnetic resonance frequency.[7] The frequency dependence of Faraday rotation shown in Fig. 9 is determined by two effects. First the guide wavelength is not linearly related to the vacuum wavelength as can be

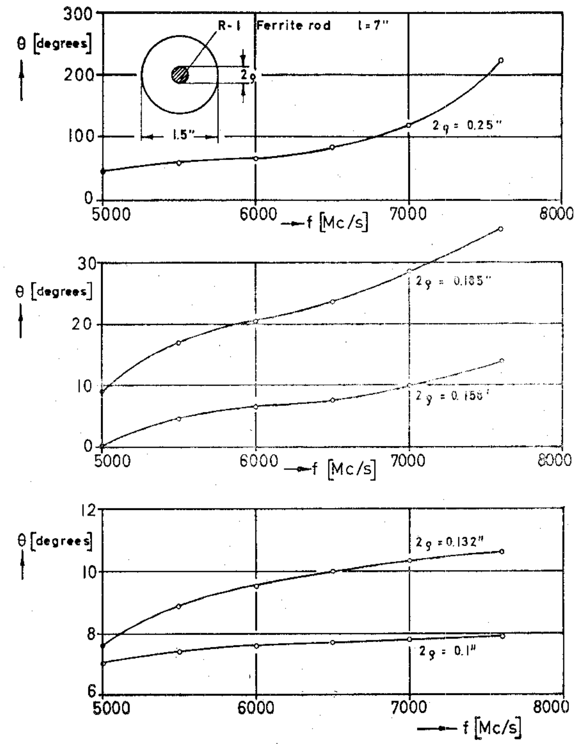

Fig. 8—Faraday rotation of a TE_{11}-wave propagating through a round waveguide containing an axial $R-1$ ferrite rod, as function of the frequency for various diameters of the ferrite rod. $H_{dc} = 26$ oersteds.

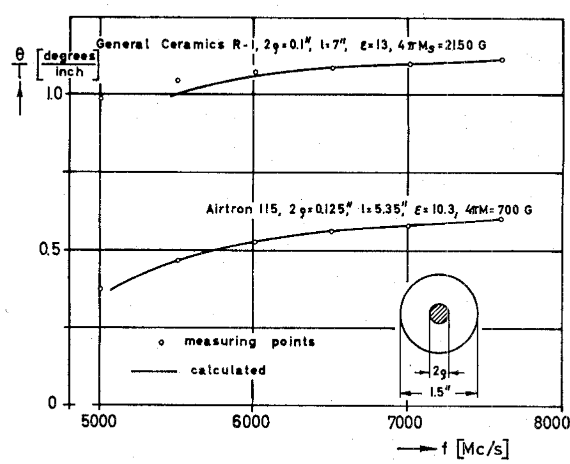

Fig. 9—Faraday rotation per unit length of a TE_{11}-wave propagating through a round waveguide containing a thin axial ferrite rod, as function of the frequency, measured and calculated by first-order perturbation method.

seen from Fig. 10. Secondly Fig. 11 shows that for a ferrite rod of the material $R-1$ the measuring frequencies are too close to ferromagnetic resonance as to neglect $\omega_{\text{res}} \ll \omega$ and apply the approximation (16). With reference to Fig. 9 the results found for the non-saturated Airtron-ferrite confirm Rado's theory on non-saturated materials.[12,13]

With increasing diameter of the ferrite rod the elec-

[12] G. T. Rado, "Theory on the microwave permeability tensor and Faraday effect in non-saturated ferromagnetic materials," *Phys. Rev.*, vol. 89, p. 529; 1953.

[13] R. C. LeCraw and E. G. Spencer, "Tensor permeabilities of ferrites below magnetic saturation," 1956 IRE Convention Record, pt. 5, pp. 66–74.

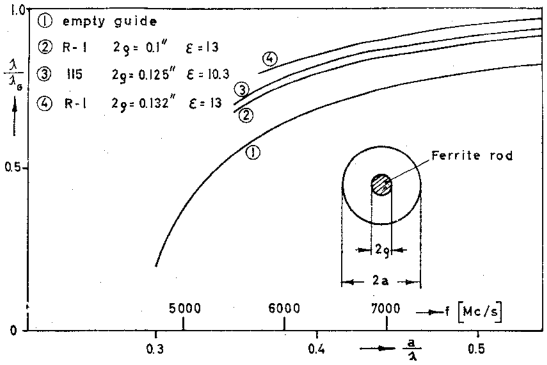

Fig. 10—Guide wavelength of the TE_{11}-wave in a round guide containing an axial dielectric rod, calculated by first-order perturbation method. λ = wavelength in free space. (Frequency scale for a waveguide diameter $2a = 1.5$ inches.)

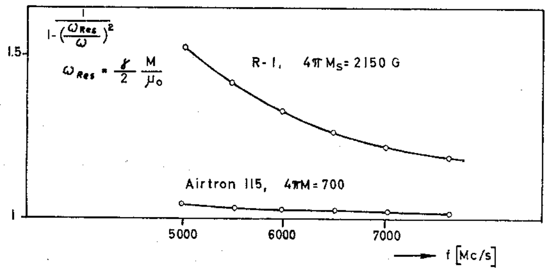

Fig. 11—Frequency dependence of the off-diagonal component K for General Ceramics $R-1$ ferrite and Airtron 155 ferrite.

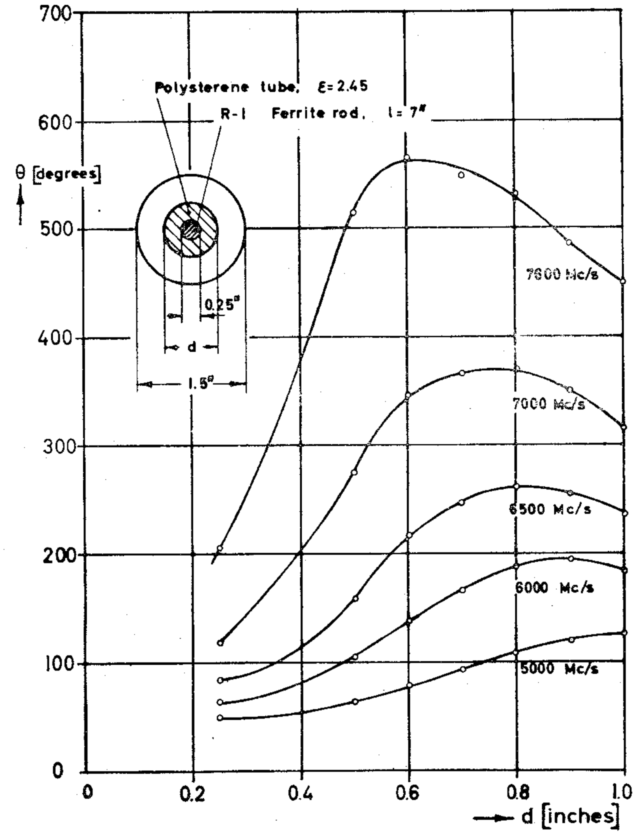

Fig. 12—Faraday rotation of a TE_{11}-wave propagating at various frequencies through a round waveguide containing an axial $R-1$ ferrite rod embedded in a polysterene tube, as function of the tube diameter. $H_{dc} = 25$ oersteds.

tromagnetic field becomes concentrated in the ferrite, as described above. This effect may be enlarged by surrounding the ferrite with a suitable dielectric tube. Achieving optimum Faraday rotation is of some practical importance if the magnetizing field cannot be strengthened without considerable expense, *e.g.*, as with high-speed steering elements. Furthermore, in this case one is restricted to long thin rods of ferrite in order to keep the demagnetization factor as small as possible.

Rowen[14] observed an enhancement of the Faraday-rotation for a partially ferrite loaded guide if the remainder of the guide is filled with a medium of high permittivity. For ferrite rods of small diameters this is confirmed by the result of our perturbation calculations (Fig. 1). According to observations of Fox, Miller and Weiss,[15] the rotation increases or decreases with increasing permittivity of the surrounding medium depending on the diameter of the ferrite rod. For larger diameters most of the electromagnetic energy is concentrated in the ferrite. Therefore, with increasing permittivity of the surrounding medium the ratio of energy in the ferrite to the total guide energy decreases and this may result in a decrease of rotation. Analogous considerations should hold if the ferrite rod is embedded in dielectric tubes of various outer diameters. For a given permittivity and frequency, the ratio of energy in the ferrite to the guide energy, and therefore, the rotation, are expected to increase up to a certain tube diameter and to decrease again if the tube diameter increases further. This is confirmed by corresponding measurements reproduced in Fig. 12. The results indicate that there is always an optimum tube diameter for which the rotation is maximum. For higher frequencies this maximum shifts to smaller tube diameters. For polysterene tubes ($\epsilon = 2.45$) a factor $2 \cdots 3$ in rotation could be won.

In Fig. 13 the optimum values of Fig. 12 have been compared with the mathematically treatable case that the ferrite rod is embedded in a dielectric of the same permittivity filling the waveguide. In this case no RF-energy is concentrated in the ferrite, because the permittivities are equal and the permeabilities differ only slightly. The comparison with the optimum rotation values of the partially loaded guide (Fig. 12), which are up to three times larger, indicates the enormous influence of the field concentration in the ferrite. The important result for practical applications is the fact that with a dielectric tube of suitable diameter, even with the relatively small permittivity $\epsilon = 2.4$, higher values of rotation can be achieved than with a dielectric of the permittivity 13 filling the cross section of the waveguide. Finally one may consider also the impedance match problem, which is much simpler to overcome in the first case.

[14] J. H. Rowen, "Ferrites in microwave applications," *Bell Sys. Tech. J.*, vol. 32, pp. 1133–1369; 1953.

[15] A. G. Fox, S. E. Miller, and M. T. Weiss, "Behavior and applications of ferrites in the microwave region," *Bell Sys. Tech. J.*, vol. 34, pp. 5–103; 1955.

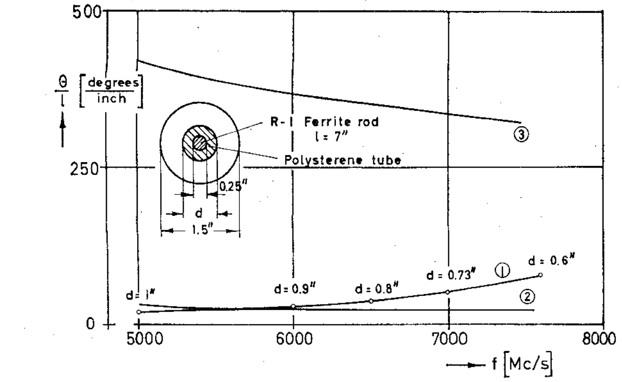

Fig. 13—Faraday rotation per unit length of a TE_{11}-wave propagating through a round waveguide
1) containing an axial ferrite rod surrounded by a dielectric tube of optimum diameter (see Fig. 12).
2) for the same ferrite rod embedded in a dielectric of equal permittivity filling the waveguide.
3) filled with ferrite.

Appendix I

If one introduces the field components (2) into formula (1) the following integrals result, which can be evaluated elementarily:

$$\int J_1^2\left(s_{11}\frac{r}{a}\right) r dr$$

$$= \frac{1}{2}\left(\frac{a}{s_{11}}\right)^2 x^2\{J_1^2(x) - J_0(x)J_2(x)\}\bigg|_{x=s_{11}(r/a)}$$

$$= \frac{1}{2}\left(\frac{a}{s_{11}}\right)^2 \{(x^2-1)J_1^2(x) + x^2 J_1'^2(x)\}\bigg|_{x=s_{11}(r/a)}$$

$$\int J_1^1\left(s_{11}\frac{r}{a}\right)\frac{J_1\left(s_{11}\frac{r}{a}\right)}{s_{11}\frac{r}{a}} r dr = \frac{1}{2}\left(\frac{a}{s_{11}}\right)^2 J_1^2\left(s_{11}\frac{r}{a}\right)$$

$$\int\left\{\frac{J_1^2\left(s_{11}\frac{r}{a}\right)}{\left(s_{11}\frac{r}{a}\right)^2} + J_1'^2\left(s_{11}\frac{r}{a}\right)\right\} r dr$$

$$= \int J_0^2\left(s_{11}\frac{r}{a}\right) r dr - 2\frac{a}{s_{11}}\int J_1\left(s_{11}\frac{r}{a}\right) J_1'\left(s_{11}\frac{r}{a}\right) dr$$

$$= \frac{1}{2}\left(\frac{a}{s_{11}}\right)^2 \{(x^2-2)J_1^2(x) + x^2 J_0^2(x)\}\bigg|_{x=s_{11}(r/a)}$$

$$= \frac{1}{2}\left(\frac{a}{s_{11}}\right)^2 \{(x^2-1)J_1^2(x) + x^2 J_1'^2(x) + 2xJ_1(x)J_1'(x)\}\bigg|_{x=s_{11}(r/a)}.$$

Appendix II

The propagation constants k_G^- and k_G^+ of the two circularly polarized waves traveling through the ferrite loaded waveguide section

$$k_G^{\mp} = k_G\left(1 + \frac{\Delta k_G^{\mp}}{k_G}\right) \tag{17}$$

can be calculated from (7) in first order approximation with the aid of the integrals given in Appendix I. In the following, the results are collated for the structures treated above.

First for the ferrite filled guide one gets

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{2}(\mu_1 - 1) \pm \frac{1}{s_{11}^2 - 1}\kappa. \tag{18}$$

In the second case of an axial ferrite rod of the radius ρ embedded in a dielectric of equal permittivity one has

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{2}\frac{1}{(s_{11}^2-1)J_1^2(s_{11})} \cdot\{(u_1-1)[(x^2-1)J_1^2(x) + x^2J_1'^2(x) + 2xJ_1(x)J_1'(x)] \pm 2\kappa J_1^2(x)\}_{x=s_{11}(\rho/a)} \tag{19}$$

which result is identical with that found by Suhl and Walker[3]; it simplifies for $\rho \ll a$ to

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{4}\frac{s_{11}^2}{(s_{11}^2-1)J_1^2(s_{11})}\left(\frac{\rho}{a}\right)^2\{(\mu_1-1)\pm\kappa\} \tag{20}$$

and for $\rho \approx a$ to

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{2}(\mu_1-1) \pm \frac{1}{s_{11}^2-1}\kappa - \frac{1}{s_{11}^2-1}(\mu_1-1) \cdot\left\{\frac{a-\rho}{a} + \frac{1}{2}\left(\frac{a-\rho}{a}\right)^2\right\} \mp \kappa\left(\frac{a-\rho}{a}\right)^2. \tag{21}$$

The result has been written in the form

$$\left(\frac{\Delta k_G}{k_G}\right)_{\substack{\text{ferrite}\\\text{tube}}} = \left(\frac{\Delta k_G}{k_G}\right)_{\substack{\text{ferrite filled}\\\text{wave guide}}} - \left(\frac{\Delta k_G}{k_G}\right)_{\substack{\text{ferrite}\\\text{rod}}}. \tag{22}$$

Because of this relation the above results also include the complementary problem of a ferrite tube adjoining the wall of the guide, the remainder of which is filled with a dielectric of the same permittivity as the ferrite. The corresponding limiting cases of ferrite tubes of very small or large wall thickness are covered by the formulas given above.

If the ferrite and the surrounding dielectric have different permittivities ϵ_F and ϵ, the application of the first-order perturbation method is restricted to small or large diameters of the ferrite rod or dielectric rod, respectively. For a thin ferrite pencil ($\rho \ll a$) from (7) one again gets (20). In (17) the propagation constant k_G of the unmagnetized structure can be found also by first order perturbation method[5] as

$$k_G = k_G^{(\epsilon)}\left\{1 + \frac{1}{4}\frac{s_{11}^2}{(s_{11}^2-1)J_1^2(s_{11})}(\epsilon_F - \epsilon)\left(\frac{k}{k_G^{(\epsilon)}}\right)^2\left(\frac{\rho}{a}\right)^2\right\}. \quad (23)$$

Specializing (20) and (23) to $\epsilon = 1$ and combining them with (17) we obtain the same result as Suhl and Walker[3] [(11) p. 1142] who treated the problem of a thin ferrite rod in an air-filled guide in a more general way. Clarricoats[16] has completed the result by considering the case of a ferrite with loss in first order approximation.

If on the other hand the radius of the ferrite rod approaches the radius of the guide, (21) holds with[5]

$$k_G = k_G^{(\epsilon_F)}\left\{1 - \frac{1}{s_{11}^2-1}(\epsilon_F - \epsilon)\left(\frac{k}{k_G^{(\epsilon_F)}}\right)^2\frac{a-\rho}{a}\right\}.$$

Finally the two complementary cases can be treated.

[16] P. I. B. Clarricoats, "Some properties of circular wave guides containing ferrites," *Proc. IEE*, vol. 104, B Suppl., pp. 286–295; 1957.

For the thin walled ferrite tube ($\rho \approx a$) adjoining the guide one gets

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{s_{11}^2-1}(\mu_1 - 1)\frac{a-\rho}{\rho} \pm \kappa\left(\frac{a-\rho}{a}\right)^2$$

with

$$k_G = k_G^{(\epsilon)}\left\{1 - \frac{1}{s_{11}^2-1}(\epsilon - \epsilon_F)\left(\frac{k}{k_G^{(\epsilon)}}\right)^2\frac{a-\rho}{a}\right\}$$

and for the wave guide filled with ferrite and an axial dielectric rod ($\rho \ll a$)

$$\frac{\Delta k_G^{\mp}}{k_G} = \frac{1}{2}(\mu_1 - 1) \pm \frac{1}{s_{11}^2-1}\kappa - \frac{1}{4}\frac{s_{11}^2}{(s_{11}^2-1)J_1^2(s_{11})}\{(\mu_1-1)\pm\kappa\}\left(\frac{\rho}{a}\right)^2$$

with

$$k_G = k_G^{(\epsilon_F)}\left\{1 + \frac{1}{4}\frac{s_{11}^2}{(s_{11}^2-1)J_1^2(s_{11})}\cdot(\epsilon - \epsilon_F)\left(\frac{k}{k_G^{(\epsilon_F)}}\right)^2\left(\frac{\rho}{a}\right)^2\right\}.$$

Magnified and Squared VSWR Responses for Microwave Reflection Coefficient Measurements*

R. W. BEATTY†

Summary—**In conventional microwave impedance measuring instruments, the measured ratio of maximum to minimum detector signal level is ideally equal to the voltage standing-wave ratio (VSWR) of the termination. In this paper, it is shown how radically different types of response are obtainable in which the observed ratio may approximately equal the square of the VSWR or may be magnified any desired amount. Theory is given enabling accurate measurements by interesting techniques. Accuracies of 0.1 per cent in VSWR to 2.0 have been achieved using magnified response techniques.**

Introduction

In most microwave impedance measuring instruments, such as the idealized slotted line, the resonance line, and rotary standing-wave indicators, the ratio of the maximum to the minimum amplitude of the output to the detector is ideally equal to the voltage standing-wave ratio (VSWR) of the termination subjected to measurement.

* Manuscript received by the PGMTT, February 20, 1959.
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Other radically different types of response are obtainable. The two responses to be discussed in this paper have been called magnified and squared VSWR responses for reasons which will become apparent.

A simplified explanation will first be given, followed by a more complete mathematical description.

The differences among responses are shown in Fig. 1, three response curves calculated for the same termination.

Simplified Explanations

Squared VSWR Response

A simplified explanation can be given for one system yielding squared VSWR response. Other systems which have been devised apparently do not permit simplified explanations and will not be thoroughly analyzed. Enough theory will be given however, to permit their use as measurement systems.

The system shown in the diagram in Fig. 2 consists